

Improving $\alpha_{\text{QED}}(M_Z^2)$ and the charm mass by analytic continuation

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Abstract. The standard determination of the QED coupling on the Z pole is performed using the latest available data for R . The direct application of analytic continuation techniques is found not to improve the accuracy of the value of $\alpha(M_Z^2)$. However they help to resolve an ambiguity in the values of R in the energy region $\sqrt{s} \lesssim 2$ GeV, which, in turn, reduces the uncertainty in $\alpha(M_Z^2)$. Moreover, they provide a sensitive determination of the mass of the charm quark. The favoured solution, which uses the inclusive data for R for $\sqrt{s} \lesssim 2$ GeV, has a pole mass $m_c = 1.33 - 1.40$ GeV and $\alpha^{-1}(M_Z^2) = 128.972 \pm 0.026$; whereas if the sum of the exclusive channels is used to determine R in this region, we find $\alpha^{-1}(M_Z^2) = 128.941 \pm 0.029$.

1 Introduction

The value of the QED coupling at the Z boson mass, $\alpha(M_Z^2)$, is the poorest known of the three parameters necessary to define the standard electroweak model, which, for example, may be taken to be G_F, M_Z and $\alpha(M_Z^2)$. The value of $\alpha(M_Z^2)$ is obtained from

$$\alpha^{-1} \equiv \alpha(0)^{-1} = 137.03599976(50) \quad (1)$$

using the relation

$$\alpha(s)^{-1} = \left(1 - \Delta\alpha_{\text{lep}}(s) - \Delta\alpha_{\text{had}}^{(5)}(s) - \Delta\alpha^{\text{top}}(s)\right) \alpha^{-1}, \quad (2)$$

where the leptonic contribution to the running of α is known to 3 loops [1]

$$\Delta\alpha_{\text{lep}}(M_Z^2) = 314.98 \times 10^{-4}. \quad (3)$$

From now on we omit the superscript (5) on $\Delta\alpha_{\text{had}}$ and assume that it corresponds to five flavours. We will include the contribution of the sixth flavour, $\Delta\alpha^{\text{top}}(M_Z^2) = -0.76 \times 10^{-4}$, at the end. To determine the hadronic contribution it is traditional to evaluate

$$\Delta\alpha_{\text{had}}(s) = -\frac{\alpha s}{3\pi} P \int_{4m_\pi^2}^{\infty} \frac{R(s') ds'}{s'(s' - s)} \quad (4)$$

at $s = M_Z^2$, where $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$.

The main uncertainty in the calculation of $\Delta\alpha_{\text{had}}$ comes from the lack of precise knowledge of $R(s')$ in the

energy region $1.5 \lesssim \sqrt{s'} \lesssim 3$ GeV, see Fig. 1. In the upper half of this interval the situation has recently improved with the new (preliminary) BES-II measurements [2]. Nevertheless there remains a major problem due to the discrepancy between the inclusive measurements of $e^+e^- \rightarrow \text{hadrons}$ and the value of the cross section deduced from the sum of all the exclusive hadronic channels ($e^+e^- \rightarrow 2\pi, 3\pi, \dots, K\bar{K}, \dots$), see Fig. 1.

Recently dispersion relation (4) has been re-evaluated at $s = M_Z^2$ [3,4], incorporating the new BES-II data for $R(s')$. In Sect. 2 we give the details of the determination of [3] and, in particular, expose the dilemma with the input values of $R(s')$ in the region $\sqrt{s'} \lesssim 2$ GeV. In Sect. 3, following Jegerlehner [5], we describe an attempt to better determine $\Delta\alpha_{\text{had}}(M_Z^2)$ by evaluating dispersion relation (4) in the space-like region, at $s = -s_0$ say, and then using perturbative QCD to analytically continue from $s = -s_0 \rightarrow -M_Z^2 \rightarrow M_Z^2$. Although this procedure is found to reduce the error associated with the data for $R(s')$, it is more than compensated by the uncertainties in the analytic continuation coming from the choice of the mass of the charm quark and the QCD scale.

However analytic continuation offers the possibility to resolve the dilemma in the data for $R(s')$ in the region $\sqrt{s'} \lesssim 2$ GeV (see Sect. 4), and to give a reasonably accurate determination of the pole mass m_c of the charm quark (see Sect. 5). Clearly a resolution of the dilemma will improve the direct determination of $\Delta\alpha_{\text{had}}(M_Z^2)$ obtained by evaluating (4) at $s = M_Z^2$. In Sect. 6 we present our conclusions.

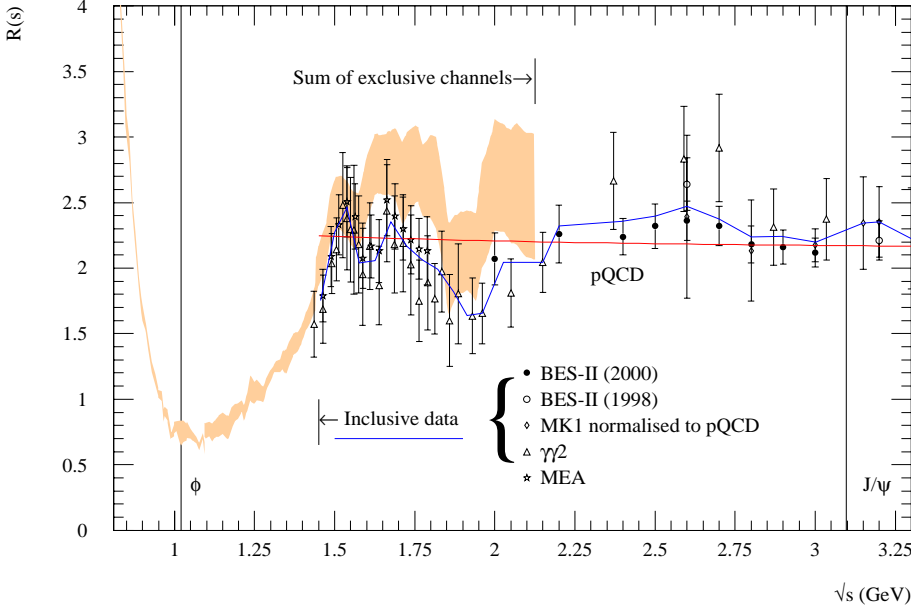


Fig. 1. The quantity $R(s)$ versus \sqrt{s} in the critical low energy interval, $\sqrt{s} \lesssim 3\text{GeV}$. The band below $\sqrt{s} = 2.125\text{GeV}$ illustrates the bounds of the summed exclusive channels. The inclusive data are explicitly plotted, and above $\sqrt{s} = 1.46\text{GeV}$ the curve shows the central value of their interpolation. In the overlapping interval there is a distinct discrepancy between the two (in principle) complementary measurements. The central perturbative QCD prediction at $\mathcal{O}(\alpha_S^3)$ is plotted through the inclusive region for comparison. Finally, the vertical lines denote the central positions of the ϕ and J/ψ resonances. (See the note added in proof for the final BES measurements [25])

Table 1. A detailed breakdown of the individual exclusive channel contributions to $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$. The dominant contribution arises from the $e^+e^- \rightarrow \pi^+\pi^-$, and the next most significant contributions are obtained from $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$ and $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$, depicted in Fig. 4. The channels marked with ⁽¹⁾ have been corrected for missing modes. The channel highlighted by ⁽²⁾ has had the $\eta \rightarrow 3\pi$ contribution subtracted. Those modes marked by ⁽³⁾ have their contributions deduced from isospin relations. The modes described in ⁽⁴⁾ are deduced from the ‘partially’ inclusive measurements of $e^+e^- \rightarrow K_S^0 + X$, with modes explicitly included elsewhere subtracted. We have checked the contributions to the cross-section from each annihilation channel with the detailed decomposition given in [13], and find excellent agreement between the two evaluations

Final state	$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) \cdot 10^4$ $2m_\pi - 1.46\text{GeV}$	$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) \cdot 10^4$ $1.46 - 1.9\text{GeV}$
$\pi^+\pi^-$	33.93 ± 0.52	0.17 ± 0.06
$\pi^+\pi^-\pi^0$	0.30 ± 0.04	0.17 ± 0.05
$\pi^+\pi^-\pi^0\pi^0$	2.00 ± 0.08	2.99 ± 0.31
$\omega \pi^0$ ⁽¹⁾	0.12 ± 0.02	0.04 ± 0.01
$\pi^+\pi^-\pi^+\pi^-$	1.45 ± 0.05	2.29 ± 0.09
$\pi^+\pi^-\pi^+\pi^-\pi^0$	0.09 ± 0.04	0.70 ± 0.25
$\pi^+\pi^-\pi^0\pi^0\pi^0$ ⁽³⁾	0.04 ± 0.05	0.33 ± 0.22
$\omega \pi^+\pi^-$ ⁽¹⁾		0.02 ± 0.00
$\pi^+\pi^-\pi^+\pi^-\pi^+\pi^-$		0.05 ± 0.02
$\pi^+\pi^-\pi^+\pi^-\pi^0\pi^0$	0.02 ± 0.01	0.82 ± 0.09
$\pi^+\pi^-\pi^0\pi^0\pi^0\pi^0$ ⁽³⁾	0.01 ± 0.01	0.61 ± 0.61
$\eta \pi^+\pi^-$ ⁽²⁾	0.02 ± 0.02	0.12 ± 0.04
K^+K^-	0.53 ± 0.05	0.16 ± 0.02
$K_S^0 K_L^0$	0.15 ± 0.11	0.04 ± 0.02
$K_S^0 K^+\pi^-(K_L^0 K^-\pi^+)$ ⁽³⁾	0.03 ± 0.01	0.28 ± 0.05
$K^+K^-\pi^0$		0.10 ± 0.07
$K_S^0 K_L^0 \pi^0$ ⁽³⁾		0.10 ± 0.07
$K\bar{K}\pi\pi$ ⁽⁴⁾	0.01 ± 0.25	1.04 ± 0.67
Sum of contributions	38.76 ± 0.79	10.32 ± 1.06

2 Direct determination of $\Delta\alpha_{\text{had}}(M_Z^2)$

In this section we give the details of the recent determination¹ of $\Delta\alpha_{\text{had}}(M_Z^2)$ that was presented in [3]. We evaluated dispersion relation (4) at $s = M_Z^2$ using the experimental data [2, 6–9] for $R(s')$ in the intervals $2m_\pi < \sqrt{s'} < 2.8\text{ GeV}$ and $3.74 < \sqrt{s'} < 5\text{ GeV}$, together with the $J/\psi, \psi'$ and Υ resonance contributions. In the remaining regions ($2.8 < \sqrt{s'} < 3.74$ and $\sqrt{s'} > 5\text{ GeV}$) we calculate $R(s')$ from perturbative QCD using the two-loop expression with the m_c and m_b quark masses included and the massless three-loop expression [10] calculated in the $\overline{\text{MS}}$ renormalization scheme². We estimate the ‘perturbative’ error on $R(s')$ by allowing m_c, m_b, M_Z to vary within the uncertainties quoted in [12], by taking $\alpha_S(M_Z^2) = 0.119 \pm 0.002$ and by varying the scale of $\alpha_S(cs)$ in the range $0.25 < c < 4$.

The errors on the ‘data’ values of $R(s')$ are calculated using a correlated χ^2 minimization to combine the different data sets, as described in detail in [13]. The data, together with the error band used in the $3.74 < \sqrt{s'} < 5\text{ GeV}$ interval, are shown in Fig. 2. For $\sqrt{s'} < 1.46\text{ GeV}$ the sum of the data for the exclusive channels is used to compute $R(s')$, see Table 1. Recently there have been improvements in our knowledge of the exclusive channels. This can be seen, for example, in the data [7] for the 2π channel shown in Fig. 3, or the data [8] for the 4π channel shown in Fig. 4.

For $\sqrt{s'} > 1.46\text{ GeV}$ we also have inclusive measurements of $R(s')$. These differ significantly from the sum of

¹ A correction to the analysis of [3] shifts the value of $\Delta\alpha_{\text{had}}$ by 0.44×10^{-4}

² The uncertainty due to using a different scheme may be estimated to be of the order of the $\mathcal{O}(\alpha_S^4)$ correction, $3\Sigma e_q^2 r_3 (\alpha_S/\pi)^4$. We may take $r_3 = -128$ [11] which leads to a negligible uncertainty in $R(s')$

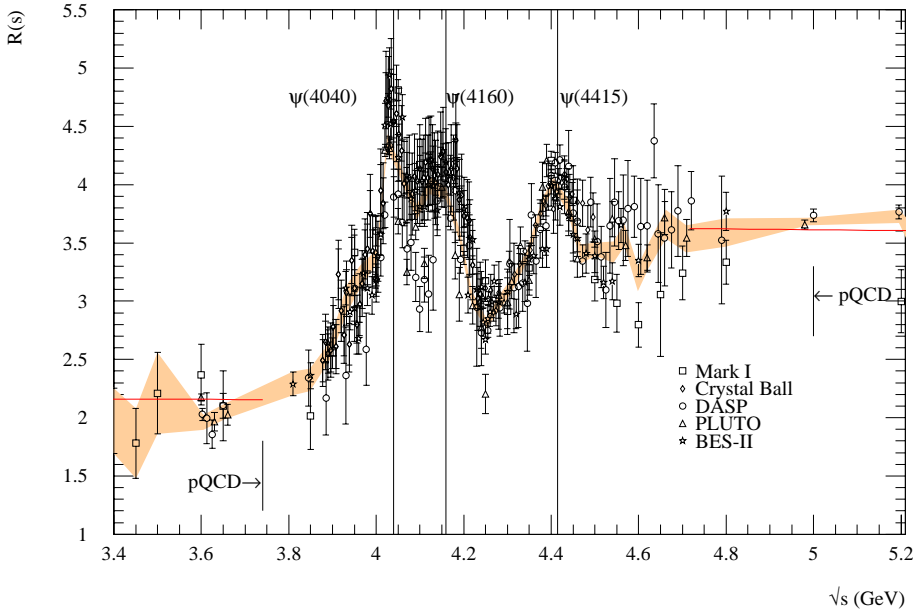


Fig. 2. The quantity $R(s)$ in the vicinity of the charm threshold $3.74 \lesssim \sqrt{s} \lesssim 5\text{GeV}$. The Mark I, DASP and PLUTO data have been scaled by factors of 0.84, 0.88 and 0.95 so as to agree with the perturbative QCD prediction in the continuum regions safely above and beneath threshold. To guide the eye, vertical lines denoting the position of the $\psi(4040)$, $\psi(4160)$ and $\psi(4415)$ resonance centres have been superimposed. The band illustrates the interpolation derived from the compilation of the (rescaled) data. The perturbative prediction for R to $\mathcal{O}(\alpha_s^3)$ is depicted in the continuum. The evaluations of $\Delta\alpha_{\text{had}}$ in this work use the perturbative prediction for $R(s)$ in the regions $2.8 < \sqrt{s} < 3.74\text{ GeV}$ and $\sqrt{s} > 5\text{ GeV}$

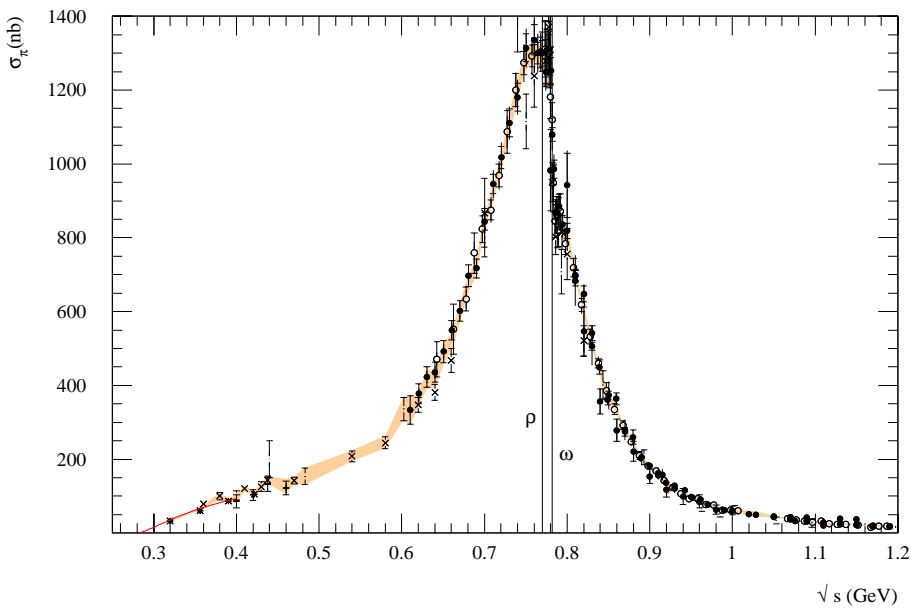


Fig. 3. The cross-section for pion pair production, $\sigma_{\pi\pi}(s)$, versus \sqrt{s} around the ρ -resonance region, $2m_\pi < \sqrt{s} \lesssim 1\text{GeV}$. The data [7] include the recent, accurate results from Novosibirsk. The band illustrates the spread of uncertainty about a central value interpolated from the data compilation. The line at low energies shows the chiral expansion of the two pion cross-section [14]

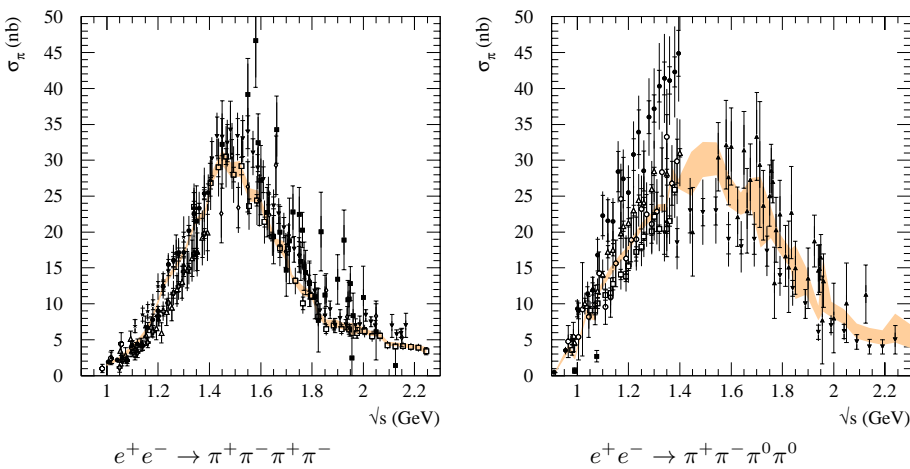


Fig. 4. The cross-section (in nb) of the four pion channels in e^+e^- annihilation. The bands again show the interpolation through the data [8,9]

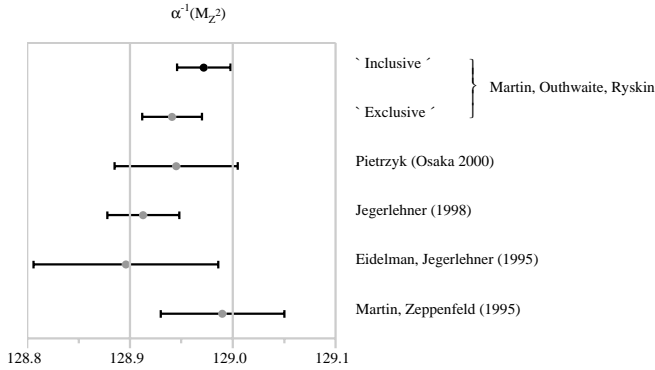


Fig. 5. Our determinations of $\alpha^{-1}(M_Z^2)$ compared to two recent other determinations [4,5], as well as two much earlier evaluations [15,16]

the exclusive channels, see Fig. 1. This poses a dilemma. The new (preliminary) BES-II data [2], which extend down to $\sqrt{s'} = 2$ GeV, appear to match better to the inclusive measurements, but the distinction is not conclusive. We therefore, throughout this paper, take two alternative choices of the data in the interval $1.46 < \sqrt{s'} < 1.9$ GeV. We first use the inclusive data and then we repeat the analysis using the exclusive data (with the error band shown in Fig. 1). For simplicity, we refer to these as the ‘inclusive’ and ‘exclusive’ data choices. In the later sections of this paper we study ways to resolve this dilemma and we present evidence which favours the ‘inclusive’ behaviour of $R(s')$ in this interval. In Table 2 we list the contributions to the dispersion relation (4) from specific $\sqrt{s'}$ intervals for both the above choices of data. In the Table we also include the corresponding values of $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ and $\alpha^{-1}(M_Z^2)$. We see that the ambiguity in the input for $R(s')$ in the region $\sqrt{s'} \lesssim 2$ GeV *itself* leads to an uncertainty of the size of the quoted errors on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$. We attempt to resolve this ambiguity in Sect. 4.

The values that we obtain for $\alpha^{-1}(M_Z^2)$ are compared with other recent determinations in Fig. 5. We also include on this plot two 1994-5 determinations in order to gain some insight. First, we show the value obtained by Martin and Zeppenfeld [15] which made use of perturbative QCD, as has become common practice, and which used ‘inclusive’ data for $\sqrt{s'} > 1.46$ GeV and rescaled data in the charm resonance region. Second, we show the value of Eidelman and Jegerlehner [16] which was obtained using data in all intervals, and hence the larger errors. For interest, we compare the individual contributions and errors of our present ‘inclusive’ determination with those of the 1995 analysis of Eidelman and Jegerlehner in Table 3.

In Fig. 6 we show the χ^2 profiles³ obtained using the ‘inclusive’ and ‘exclusive’ determinations of the QED coupling $\alpha(M_Z^2)$ in fits to the latest compilation of electroweak data for different values of the mass of the (standard Model) Higgs boson. We see that the minimum obtained using the ‘inclusive’ value, $\alpha(M_Z^2) = 1/128.972$, is close to the LEP2 bound on the Higgs mass.

³ We thank Martin Grünewald for making this plot

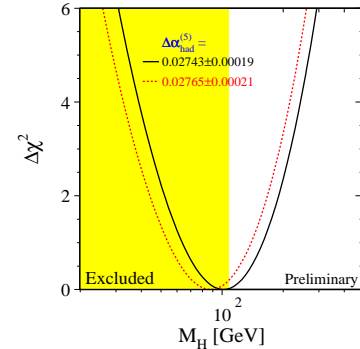


Fig. 6. χ^2 fit as a function of the standard model Higgs mass, M_H , to the latest compilation of electroweak data, obtained using the ‘inclusive’ and ‘exclusive’ determinations of $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) \cdot 10^4$ of 274.3 (continuous curve) and 276.5 (dashed curve). The shaded zone to the left illustrates the energy interval where the Higgs has been excluded by direct searches at LEP2

3 Analytic continuation in the space-like region

There have been several studies [17,5] of analytic behaviour in the complex s -plane in attempts to reduce the dependence of the determination of $\Delta\alpha_{\text{had}}(M_Z^2)$ on the observed values of R in the region in which it is poorly known. These techniques have been reviewed by Jegerlehner [5]. He concludes that it is difficult to reduce the error on $\Delta\alpha_{\text{had}}$ due to the data in this way. He advocates the following analytic continuation method to determine $\Delta\alpha_{\text{had}}(M_Z^2)$. First, evaluate (4) for space-like $s = -s_0$ and then use perturbative QCD to continue to $s = -M_Z^2$, that is

$$\Delta\alpha_{\text{had}}(-M_Z^2) = [\Delta\alpha_{\text{had}}(-M_Z^2) - \Delta\alpha_{\text{had}}(-s_0)]^{\text{QCD}} + \Delta\alpha_{\text{had}}(-s_0)^{\text{data}} \quad (5)$$

where s_0 is chosen sufficiently large ($\sqrt{s_0} \gtrsim 2$ GeV) for the QCD contribution in square brackets to be known accurately⁴, such that the error in $\Delta\alpha_{\text{had}}(-M_Z^2)$ dominantly reflects the error in the data for $R(s')$. The error associated with the final continuation round the semicircle to $\Delta\alpha_{\text{had}}(M_Z^2)$ is negligible

$$\Delta\alpha_{\text{had}}(M_Z^2) = \Delta\alpha_{\text{had}}(-M_Z^2) + (0.42 \pm 0.02) \times 10^{-4}. \quad (6)$$

Jegerlehner [5] chose $\sqrt{s_0} = 2.5$ GeV and found⁵

$$\Delta\alpha_{\text{had}}(M_Z^2) = (277.82 \pm 2.54) \times 10^{-4} \quad (7)$$

where the error was entirely attributed to that for the contribution $\Delta\alpha_{\text{had}}(-s_0)^{\text{data}}$ to (5).

We will examine this proposal below. In particular we will investigate whether it is possible to develop this technique *either* to select between the inclusive/exclusive $R(s')$

⁴ Previous studies [18] had indicated how large s_0 had to be to avoid uncertainties due to parton condensate contributions

⁵ The recent BES-II data [2] were not available for the analysis of [5]

Table 2. The individual contributions to the hadronic component of the shift in fine structure constant, $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) \cdot 10^4$. The upper (lower) error in the result labelled ^a corresponds to the 2π (remaining) exclusive channels. Contributions labelled with superscripts ^b, ^c and ^d have common error sources which are added linearly. Remaining errors are added in quadrature

\sqrt{s} interval (GeV)	$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) \cdot 10^4$ contribution	Origin of contribution
$2m_\pi - 1.46^a$	$38.76 \pm \left\{ \begin{array}{l} 0.52 \\ 0.60^b \end{array} \right\}$	Pion form factor data
1.46 - 1.90	$\left\{ \begin{array}{l} 8.62 \pm 0.60^c \\ 10.32 \pm 1.06^b \end{array} \right\}$	$\left\{ \begin{array}{l} \text{Inclusive data} \\ \text{Exclusive summation} \end{array} \right\}$
1.90 - 2.80	$\left\{ \begin{array}{l} 13.26 \pm 0.83^c \\ 13.79 \pm 0.83 \end{array} \right\}$	$\left\{ \begin{array}{l} \text{Inclusive data} \\ \text{Exclusive summation} \end{array} \right\}$
2.80 - 3.74	9.73 ± 0.05^d	Perturbative QCD
3.74 - 5.00	15.13 ± 0.36	Charm data
5.00 - ∞	169.97 ± 0.64^d	Perturbative QCD
ω, ϕ, ψ 's, Υ 's	18.79 ± 0.58	Breit-Wigner resonances
$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) \cdot 10^4$	$\left\{ \begin{array}{l} 274.26 \pm 1.90 \\ 276.49 \pm 2.14 \end{array} \right\}$	$\left\{ \begin{array}{l} \text{Inclusive data} \\ \text{Exclusive summation} \end{array} \right\}$
$\alpha^{-1}(M_Z^2)$	$\left\{ \begin{array}{l} 128.972 \pm 0.026 \\ 128.941 \pm 0.029 \end{array} \right\}$	$\left\{ \begin{array}{l} \text{Inclusive data} \\ \text{Exclusive summation} \end{array} \right\}$

Table 3. A comparison of the individual contributions to $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) \cdot 10^4$ found in the 1995 ‘data-driven’ analysis of Eidelman and Jegerlehner [16], with those of our inclusive analysis, decomposed according to the energy intervals used in [16]

Final state	\sqrt{s} interval (GeV)	Contribution from [16]	Current evaluation
ρ	$2m_\pi - 0.81$	26.08 ± 0.68	25.32 ± 0.52
ω	0.42 - 0.81	2.93 ± 0.09	3.07 ± 0.10
ϕ	1.00 - 1.04	5.08 ± 0.14	5.08 ± 0.19
J/ψ		11.34 ± 0.82	$9.41 \pm 0.53 (+1.93=11.35)$
Υ		1.18 ± 0.08	1.22 ± 0.04
hadrons	0.81 - 1.40	13.83 ± 0.80	12.24 ± 0.54
hadrons	1.40 - 3.10	27.62 ± 4.02	$\left\{ \begin{array}{l} 1.40 - 1.46 \quad 1.21 \pm 0.07 \quad \text{Exc.} \\ 1.46 - 2.8 \quad 21.88 \pm 1.43 \quad \text{Inc.} \\ 2.8 - 3.10 \quad 3.43 \pm 0.02 \quad \text{pQCD.} \end{array} \right\}$
hadrons	3.10 - 3.60	5.82 ± 1.16	5.02 ± 0.03
hadrons	3.60 - 9.46	50.60 ± 3.33	$\left\{ \begin{array}{l} 3.60 - 3.74 \quad 1.28 \pm 0.01 \quad \text{pQCD.} \\ 3.74 - 5.0 \quad 15.13 \pm 0.36 \quad \text{Inc.} \\ 5.0 - 9.46 \quad 35.51 \pm 0.21 \quad \text{pQCD.} \end{array} \right\}$
hadrons	9.46 - 40.0	93.07 ± 3.50	91.77 ± 0.19
perturbative QCD	40.0 - ∞	42.82 ± 0.10	42.70 ± 0.24
Total	$2m_\pi - \infty$	280.37 ± 6.54	274.26 ± 1.90

data choices in the region $\sqrt{s'} \lesssim 2$ GeV, or to reduce the importance of the data contribution (and its associated error) from this domain.

Suppose, for example, we evaluate $\alpha(M_Z^2)$ from (4), (5) and (6) for a range of different values of s_0 . In principle, we should always get the same answer. If the answer varies significantly either the data for $R(s')$ is not quite correct or the theory input is deficient in some way or, more likely, it is a combination of both. The interplay between the

uncertainties in the theory and the data (that is, in the two terms on the right-hand-side of (5)) play a crucial role in this type of analysis. If it is possible to find a choice of input data, together with a physically meaningful set of theory parameters (charm mass m_c , choice of scale etc.), which give a stable value of $\alpha(M_Z^2)$ for different choices of s_0 , then it will be a powerful argument in favour of their veracity.

Indeed, imagine one extreme in which the theory contribution to (5) was known precisely; that is, there is no error associated with the term in square brackets. Then the behaviour of the variation of $\alpha(M_Z^2)$ as a function of s_0 would highlight the domain (or domains) in which the data were wrong and, moreover, specify the approximate corrections that are necessary.

In this section we evaluate $\Delta\alpha_{\text{had}}(s)$ of (4) in the space-like domain $s = -s_0$ (with $s_0 > 0$) for a range of different values of s_0 . For each s_0 we then use perturbative QCD to perform the analytic continuation to $s = M_Z^2$, as given in (5) and (6). A sample of the results for $\Delta\alpha_{\text{had}}(-s_0)$ is presented in Table 4, together with the conventional time-like evaluation of (4) at $s = +M_Z^2$. We see that the error on the space-like evaluation of $\Delta\alpha_{\text{had}}(-s_0)$ is reduced as s_0 is decreased in comparison to that for $s = \pm M_Z^2$. This reduction may be anticipated, since from the form of (4) we see that the error mainly arises from uncertainties in the data for $R(s')$ with $s' \lesssim |s_0|$.

Let us illustrate this point in more detail. If we compare the calculation of $\Delta\alpha_{\text{had}}(-s_0)^{\text{data}}$ with the direct evaluation of $\Delta\alpha_{\text{had}}(-M_Z^2)^{\text{data}}$, then essentially we make the replacement

$$\frac{M_Z^2 R(s')}{s' + M_Z^2} \simeq R(s') \quad \rightarrow \quad \frac{s_0 R(s')}{s' + s_0} \quad (8)$$

in the integrand of (4), where for simplicity we consider $s' \ll M_Z^2$. Then we add to $\Delta\alpha_{\text{had}}(-s_0)^{\text{data}}$ the QCD term $[\Delta\alpha_{\text{had}}(-M_Z^2) - \Delta\alpha_{\text{had}}(-s_0)]^{\text{QCD}}$, as in (5). That is, if we compare the analytic continuation determination, (5), of $\Delta\alpha_{\text{had}}(-M_Z^2)$ with the direct determination $\Delta\alpha_{\text{had}}(-M_Z^2)^{\text{data}}$, then effectively we make the replacement

$$R(s')^{\text{data}} \quad \rightarrow \quad \frac{s_0 R(s')^{\text{data}} + s' R(s')^{\text{QCD}}}{s_0 + s'} \quad (9)$$

for $s' \ll M_Z^2$. Thus for $s' \ll s_0$ we keep all the data, while if $s' \sim s_0$ we use pQCD to replace about half of the data, and for $s' \gg s_0$ we discard almost all the data in favour of pQCD. Thus the lower that we can take s_0 , the smaller the data contribution, and hence the smaller its contribution to the error on $\Delta\alpha_{\text{had}}(\pm M_Z^2)$.

However before we can take advantage of the reduction of the uncertainty associated with the data, we must consider the error in the perturbative QCD continuation from $s = -s_0$ to $s = -M_Z^2$. That is the error on

$$\begin{aligned} \delta(s_0) &\equiv [\Delta\alpha_{\text{had}}(-M_Z^2) - \Delta\alpha_{\text{had}}(-s_0)]^{\text{QCD}} \\ &= -4\pi\alpha \int_{-s_0}^{-M_Z^2} ds' \frac{d\Pi(s')}{ds'}, \end{aligned} \quad (10)$$

where Π , the hadronic contribution to the photon vacuum polarisation amplitude, satisfies

$$\Delta\alpha_{\text{had}}(s) = -4\pi\alpha \text{Re}\Pi(s), \quad R(s) = 12\pi \text{Im}\Pi(s). \quad (11)$$

The error associated with the remaining analytic continuation round the semicircle from $s = -M_Z^2$ to $s = M_Z^2$ is much smaller and may be neglected, see (6).

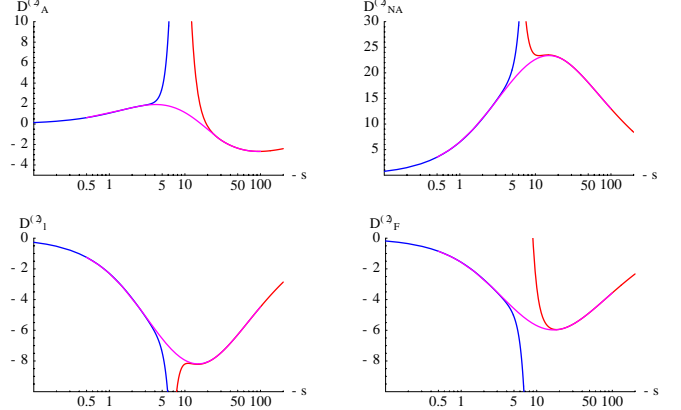


Fig. 7. Figure illustrating the efficacy of the Padé interpolation technique through threshold for $\mu = 20\text{ GeV}$ and a charm mass of $m_c = 1.4\text{ GeV}$ as a generic example. The $\mathcal{O}(\alpha_S^2)$ contribution to the Adler D -function is shown as high and low energy expansions, for (i) the pseudo-Abelian contribution containing no internal loops, (ii) the non-Abelian contributions containing triple gluon vertices, (iii) the contribution corresponding to the radiation of an internal light quark loop from a massive external quark loop, and (iv) contribution corresponding to the radiation of an internal massive quark loop, of the same mass scale as a massive external quark loop. The Padé threshold interpolation (continuous curve) becomes indistinguishable from the mass expansions away from threshold

To evaluate $\delta(s_0)$ of (10) we use the known expression for $\Pi(s)$ to $\mathcal{O}(\alpha_S^3)$. For the $\mathcal{O}(1)$ and $\mathcal{O}(\alpha_S)$ contributions we use the full analytic formula [19], which includes the dependence on the quark masses. The $\mathcal{O}(\alpha_S^2)$ contribution is evaluated in terms of the high (m_q^2/s) and low energy (s/m_q^2) expansions [20] using a (4/4) Padé interpolation⁶ for $s \sim 4m_q^2$ [21] and, finally, the massless quark limit of the $\mathcal{O}(\alpha_S^3)$ contribution is used. The expressions are valid for fixed coupling $\alpha_S(\mu^2)$. In Table 5 we show the individual contributions to $\delta(s_0)$ for a choice $m_c = 1.4\text{ GeV}$ of the pole mass of the charm quark and $\mu = 20\text{ GeV}$ of the QCD scale. Unfortunately there are appreciable uncertainties in the perturbative QCD determination of $\delta(s_0)$ arising from the sensitivity to the values taken for m_c (and m_b) and the QCD scale μ . In addition, in a recent paper Chetyrkin et al. [22] have evaluated the m^4/s^2 term in the $\mathcal{O}(\alpha_S^3)$ contribution to $R(s')$. Of course knowing just the first two terms [23, 22] in the m^2/s expansion is not sufficient to calculate the $\mathcal{O}(\alpha_S^3)$ heavy quark effect, which comes mainly from the threshold region. However knowledge of these terms enables us to estimate the typical size of the $\mathcal{O}(\alpha_S^3)$ mass contribution to be of the order of $(0.2 - 0.5) \times 10^{-4}$. In total, these ‘theoretical’ uncertainties in the QCD contribution to (5) are comparable with the error presented in Table 1 for the direct evaluation of $\Delta\alpha_{\text{had}}(M_Z^2)$. We conclude that although the error

⁶ An example of the power of the Padé interpolation is shown in Fig. 7. To calculate the $\mathcal{O}(\alpha_S^2)$ contribution to $\delta(s_0)$ we perform the appropriate integration of the Padé interpolation over the interval $s = -s_0$ to $s = -M_Z^2$

Table 4. Explicit breakdown of the contributions to $\Delta\alpha_{\text{had}}^{(5)}(s = -s_0)$ in the spacelike region for $6\text{GeV}^2 \leq s_0 \leq M_Z^2$. Again we show alternative results for the energy intervals $1.46 \leq \sqrt{s'} \leq 2.8\text{GeV}^2$ and the final sum, where the upper (lower) braced entry corresponds to the use of inclusive (exclusive) data. The perturbative contributions here were evaluated with all u, d, s, c and b flavours in their active domains, and five light quarks contributing internal loops at $\mathcal{O}(\alpha_S^2)$ and $\mathcal{O}(\alpha_S^3)$. The scale is taken as $\mu = 20\text{GeV}$, the c pole mass as 1.4GeV and the b pole mass as 4.7GeV . For convenience we show in the last column the direct evaluation of Section 2, except that here, for consistency with the space-like evaluations, we use a fixed QCD scale $\mu = 20\text{ GeV}$ and five light quarks in the internal loops. The individual errors are combined as in Table 2

$\sqrt{s'}$ interval (GeV)	$s = -6\text{GeV}^2$	$s = -15\text{GeV}^2$	$s = -50\text{GeV}^2$	$s = -M_Z^2$	$s = M_Z^2$
$2m_\pi - 0.81$	23.40 ± 0.48	24.51 ± 0.50	25.07 ± 0.51	25.31 ± 0.52	25.31 ± 0.52
$0.81 - 1.46$	11.31 ± 0.50	12.49 ± 0.56	13.14 ± 0.59	13.45 ± 0.60	13.45 ± 0.61
$1.46 - 1.9$	$\left\{ \begin{array}{l} 5.90 \pm 0.40 \\ 7.05 \pm 0.72 \end{array} \right.$	$\left\{ \begin{array}{l} 7.27 \pm 0.50 \\ 8.70 \pm 0.89 \end{array} \right.$	$\left\{ \begin{array}{l} 8.17 \pm 0.56 \\ 9.77 \pm 1.00 \end{array} \right.$	$\left\{ \begin{array}{l} 8.62 \pm 0.60 \\ 10.31 \pm 1.06 \end{array} \right.$	$\left\{ \begin{array}{l} 8.62 \pm 0.60 \\ 10.32 \pm 1.06 \end{array} \right.$
$1.9 - 2.8$	$\left\{ \begin{array}{l} 6.95 \pm 0.44 \\ 7.28 \pm 0.44 \end{array} \right.$	$\left\{ \begin{array}{l} 9.70 \pm 0.61 \\ 6.67 \pm 0.16 \end{array} \right.$	$\left\{ \begin{array}{l} 11.93 \pm 0.75 \\ 12.42 \pm 0.75 \end{array} \right.$	$\left\{ \begin{array}{l} 13.24 \pm 0.83 \\ 13.77 \pm 0.83 \end{array} \right.$	$\left\{ \begin{array}{l} 13.26 \pm 0.83 \\ 13.79 \pm 0.83 \end{array} \right.$
$2.8 - 3.74$	3.53 ± 0.01	5.68 ± 0.02	7.98 ± 0.03	9.65 ± 0.03	9.67 ± 0.03
$3.74 - 5$	3.65 ± 0.09	6.67 ± 0.16	10.92 ± 0.26	15.06 ± 0.36	15.13 ± 0.36
$5 - \infty$	6.11 ± 0.02	13.36 ± 0.09	31.38 ± 0.18	169.99 ± 0.52	170.23 ± 0.53
ω, ϕ, ψ 's, Υ 's	10.60 ± 0.27	13.37 ± 0.38	16.17 ± 0.49	18.73 ± 0.58	18.79 ± 0.58
$\Delta\alpha_{\text{had}}^{\text{data}}(-s) \cdot 10^4$	$\left\{ \begin{array}{l} 71.45 \pm 1.13 \\ 72.93 \pm 1.41 \end{array} \right.$	$\left\{ \begin{array}{l} 93.05 \pm 1.41 \\ 94.91 \pm 1.70 \end{array} \right.$	$\left\{ \begin{array}{l} 124.76 \pm 1.64 \\ 126.85 \pm 1.92 \end{array} \right.$	$\left\{ \begin{array}{l} 274.05 \pm 1.86 \\ 276.27 \pm 2.12 \end{array} \right.$	$\left\{ \begin{array}{l} 274.46 \pm 1.86 \\ 276.69 \pm 2.12 \end{array} \right.$

Table 5. The individual contributions to $\delta^{\text{QCD}}(s_0) \equiv [\Delta\alpha_{\text{had}}(-M_Z^2) - \Delta\alpha_{\text{had}}(s_0)]^{\text{QCD}}$ to $\mathcal{O}(\alpha_S^3)$ from the u, d, s and c flavours. Note that the QCD contributions in the earlier Table 4 also include the b quark

Contribution	Flavour	$s_0 = 6\text{GeV}^2$	15	25	50	100	50^2
$\mathcal{O}(1)$	u, d, s	112.02	97.83	89.92	79.19	68.46	18.61
$\mathcal{O}(\alpha_S)$		5.53	4.83	4.44	3.91	3.38	0.92
$\mathcal{O}(\alpha_S^2)$		0.69	0.39	0.25	0.10	-0.03	-0.15
$\mathcal{O}(\alpha_S^3)$		0.38	0.22	0.15	0.08	0.04	0.01
$\mathcal{O}(1)$	c	64.17	59.62	56.20	50.72	44.54	12.37
$\mathcal{O}(\alpha_S)$		5.20	4.41	3.92	3.27	2.67	0.64
$\mathcal{O}(\alpha_S^2)$		1.62	1.08	0.78	0.44	0.18	-0.10
$\mathcal{O}(\alpha_S^3)$		0.26	0.14	0.10	0.06	0.03	0.01
$\delta^{\text{QCD}}(s_0) \cdot 10^4$	u, d, s, c	189.87	168.53	155.76	137.76	119.28	32.31

on $\Delta\alpha_{\text{had}}(-s_0)$, with $s_0 = 6\text{ GeV}^2$, is considerably improved in comparison to that for the direct determination of $\Delta\alpha_{\text{had}}(M_Z^2)$, nevertheless the uncertainties in the analytic continuation from $-s_0$ to M_Z^2 means that the accuracy to which $\Delta\alpha_{\text{had}}(M_Z^2)$ is known has not been improved by the analytic continuation approach.

4 Resolution of the “inclusive-exclusive” ambiguity

We have seen that analytic continuation does not appear to allow us to reduce the uncertainty in the determination of $\Delta\alpha_{\text{had}}(M_Z^2)$. However if we turn the analysis around we have the possibility to

- (i) distinguish between the inclusive and exclusive data for $R(s)$ for $\sqrt{s} \lesssim 2\text{ GeV}$,
- (ii) constrain the value of the charm mass m_c .

To do this we study the difference between the ‘direct’ prediction for $\Delta\alpha_{\text{had}}(M_Z^2)$ (shown in the last column of Table 4) and the values obtained via the analytic continuation method of (5) and (6). Let us denote the difference of the two determinations by $d(s_0)$, that is

$$d(s_0) \equiv \Delta\alpha_{\text{had}}(M_Z^2)|_{\text{direct}} - \Delta\alpha_{\text{had}}(M_Z^2)|_{\text{anal. cont. from } s_0}. \quad (12)$$

A self-consistent analysis requires that $d(s_0) \simeq 0$ for all values of s_0 . Of course the perturbative QCD contribution depends on the value taken for the charm mass m_c and the scale μ . We therefore proceed in stages. First we remove the dependence on m_c (and m_b). We include only con-

Table 6. The discrepancy $d(s_0) \times 10^4$ of (12) for space-like evaluations at $s = -s_0$ for three different scales μ (in GeV). In the first half of the table the inclusive data for $R(s')$ are used in the region $\sqrt{s'} \lesssim 2$ GeV, whereas in the second half the exclusive data are taken

s_0 GeV ²	inclusive data			exclusive data		
	$\mu = 10$	$\mu = 20$	$\mu = 50$	$\mu = 10$	$\mu = 20$	$\mu = 50$
6	-0.31	-0.12	0.03	0.43	0.62	0.77
15	-0.16	-0.03	0.07	0.20	0.33	0.43
25	-0.13	-0.01	0.06	0.11	0.23	0.30
50	-0.11	-0.03	0.04	0.02	0.10	0.17
100	-0.07	0.00	0.04	-0.01	0.06	0.10

tributions from u, d and s quarks, and substitute for the data and resonances in the charm (and bottom) threshold regions with the values obtained from three-flavour perturbative QCD. The results for the discrepancy $d(s_0)$ are shown in Table 6 for three different choices of the scale μ . It is immediately seen that, in general, if we use the inclusive $R(s')$ data in the region $\sqrt{s'} \lesssim 2$ GeV we obtain better agreement (that is a smaller discrepancy $d(s_0)$) than if we use the exclusive data. Moreover the scale μ^2 should be representative of the interval of continuation from $s = -s_0$ to $s = -M_Z^2$, and $\mu = 20$ GeV is a reasonable choice. If we assume that the systematic discrepancy $d(s_0)$ comes from a local region $s' \simeq s_p$ then the additional contribution to the dispersion integral may be approximated by⁷

$$d(s_0) \simeq \frac{\alpha}{3\pi} \int ds' \delta(s' - s_p) \frac{R_p}{(s' + s_0)} \simeq \frac{\alpha R_p}{3\pi(s_0 + s_p)}. \quad (13)$$

In fact the differences $d(s_0)$ for the exclusive data at $\mu = 20$ GeV are well described by this simple pole form with

$$\sqrt{s_p} = 2.1 \text{ GeV}, \quad R_p = 0.8 \text{ GeV}^2. \quad (14)$$

This is consistent with the exclusive contribution being too large in the region $\sqrt{s'} \sim 2$ GeV. We may conclude the three-flavour analysis of this section favours the inclusive data for $R(s')$ for $\sqrt{s'} \lesssim 2$ GeV and, moreover, gives a remarkably consistent description with $d(s_0) \simeq 0$ for different choices of s_0 for scale choices in the region 20–50 GeV.

5 Implications for the charm mass

We now extend the ‘discrepancy’ analysis of the previous section to four-flavours and reinstate the data for $R(s')$ in the charm threshold region (that is the $J/\psi, \psi'$ and $3.74 < \sqrt{s'} < 5$ GeV). We show the results in Table 7 for

⁷ We may use the unsubtracted form of the dispersion integral since in a difference calculation the subtraction constant will cancel

a range of choices of the charm mass m_c , taking the scale $\mu = 20$ GeV. We see a systematic trend of the behaviour of $d(s_0)$ with m_c and that the choice $m_c = 1.40$ GeV gives good consistency for all s_0 if the inclusive data are used in the region $\sqrt{s'} \lesssim 2$ GeV. The numbers in brackets in Table 7 correspond to using the exclusive data up to $\sqrt{s'} \lesssim 2$ GeV. There is no choice of m_c that gives the same consistency as for the inclusive data. The optimum value appears to be $m_c = 1.34$ GeV.

The discrepancies $d(s_0)$ were fitted to the pole form (13), and the parameters (the residue R_p and pole position s_p) are given in Table 8. Again we see the inclusive data select $m_c = 1.40$ GeV (for $\mu = 20$ GeV) and that as we depart from this value the additional pole contribution is such as to compensate for the poorer choice of m_c . For the exclusive data we confirm that the value $m_c = 1.34$ GeV is optimum, but that the pole compensation for other choices of m_c is more erratic. We repeated the whole analysis for scale $\mu = 50$ GeV. The pole parameters which fit the discrepancy $d(s_0)$ in this case are also shown in Table 8 (in the last two columns). For this choice of μ , the inclusive data give $m_c = 1.33$ GeV whereas the exclusive data select $m_c = 1.26$ GeV.

Our determinations of m_c refer to the pole mass of the charm quark. However the PDG [12] gives the value of the charm mass $m_c(\mu = m_c)$ in the $\overline{\text{MS}}$ scheme, that is the running mass at scale m_c . They quote $m_c(m_c) = 1.25 \pm 0.10$ GeV, which is determined from charmonium and D meson masses. In our calculation the pole mass naturally occurs in the space-like continuation, with the ‘running’ included in the expression for the vacuum polarisation. The PDG value corresponds to a pole mass $m_c = 1.46 \pm 0.11$ GeV. We summarize the determinations⁸ in Table 9.

Again we see that the results favour the inclusive measurement of $R(s)$ in the region $\sqrt{s} \lesssim 2$ GeV. First, the inclusive data satisfy the self-consistency check $d(s_0) \simeq 0$ for different s_0 for some value of m_c , better than the exclusive data, see Table 7. Second, the prediction for the pole mass $m_c = 1.33 - 1.40$ GeV is in better agreement with PDG expectations than our prediction $m_c = 1.26 - 1.34$ GeV obtained using the exclusive data.

6 Summary

Traditionally the value of the QED coupling on the Z pole has been determined by evaluating the dispersion relation, (4), for $\Delta\alpha_{\text{had}}(s)$ at $s = M_Z^2$. In Sect. 2 we presented an updated calculation of $\Delta\alpha_{\text{had}}(M_Z^2)$ using the latest available data for $R \equiv \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$. The main uncertainty is the input for $R(s')$ in the region $1.5 \lesssim \sqrt{s'} \lesssim 3$ GeV. The new (preliminary) BES-II data [2] have improved the knowledge of $R(s')$ in the upper part of this region, so that the error on $\Delta\alpha_{\text{had}}(M_Z^2)$ is about $\pm 2 \times 10^{-4}$ corresponding to about ± 0.03 on $\alpha^{-1}(M_Z^2)$ [3].

⁸ Some years ago an analysis [24] of the moments of $R_c(s)$, obtained from $e^+e^- \rightarrow c\bar{c}$ annihilation, gave $m_c = 1.34 \pm 0.02$ GeV

Table 7. The discrepancy $d(s_0) \equiv \delta^{\text{data}}(s_0) - \delta^{\text{QCD}}(s_0)$ for a spectrum of charm pole masses and the lower QCD scale $\mu = 20\text{GeV}$. The entries (bracketed) correspond to the use of the interpolations of the inclusive (exclusive) $R(s')$ data in the region $\sqrt{s'} \lesssim 2\text{ GeV}$ of the dispersion integral (4)

$m_c(\text{GeV})$	$d(s_0 = 6\text{GeV}^2) \cdot 10^4$	$d(15) \cdot 10^4$	$d(25) \cdot 10^4$	$d(50) \cdot 10^4$	$d(100) \cdot 10^4$
1.46	0.57 (1.31)	0.36 (0.72)	0.23 (0.47)	0.10 (0.23)	0.04 (0.10)
1.44	0.39 (1.13)	0.24 (0.60)	0.16 (0.40)	0.06 (0.19)	0.03 (0.09)
1.42	0.20 (0.93)	0.12 (0.48)	0.07 (0.31)	0.02 (0.15)	0.00 (0.06)
1.40	0.00 (0.75)	0.01 (0.37)	0.00 (0.24)	-0.03 (0.10)	-0.03 (0.03)
1.38	-0.17 (0.57)	-0.09 (0.27)	-0.08 (0.16)	-0.07 (0.06)	-0.04 (0.02)
1.36	-0.37 (0.37)	-0.21 (0.15)	-0.16 (0.08)	-0.11 (0.02)	-0.06 (-0.01)
1.34	-0.57 (0.17)	-0.33 (0.03)	-0.24 (-0.00)	-0.16 (-0.03)	-0.09 (-0.03)
1.32	-0.74 (-0.01)	-0.43 (-0.07)	-0.31 (-0.07)	-0.19 (-0.06)	-0.11 (-0.05)
1.30	-0.94 (-0.20)	-0.54 (-0.18)	-0.39 (-0.15)	-0.25 (-0.12)	-0.14 (-0.08)
1.28	-1.11 (-0.38)	-0.64 (-0.28)	-0.45 (-0.21)	-0.28 (-0.15)	-0.15 (-0.09)
1.26	-1.32 (-0.58)	-0.76 (-0.40)	-0.53 (-0.29)	-0.32 (-0.19)	-0.18 (-0.12)

Table 8. The parameters (R_p, s_p) describing the simple pole fits, (13) to the residual function $d(s_0)$ for the spectrum of charm masses. The entry denoted by (**,**) corresponds to a residual sufficiently close to 0 for all s_0 to render the fitting procedure inappropriate

$m_c(\text{GeV})$	‘Inclusive’	‘Exclusive’	‘Inclusive’	‘Exclusive’
	$\mu = 20\text{GeV}$	$\mu = 20\text{GeV}$	$\mu = 50\text{GeV}$	$\mu = 50\text{GeV}$
1.46	(0.9, 6.4)	(1.8, 4.4)	(2.3, 9.4)	(3.1, 6.6)
1.44	(0.6, 5.8)	(1.5, 3.9)	(2.0, 10.0)	(2.8, 6.6)
1.42	(0.3, 4.5)	(1.1, 3.3)	(1.7, 10.7)	(2.5, 6.5)
1.40	(0.0, -4.6)	(0.8, 2.4)	(1.3, 10.8)	(2.1, 6.1)
1.38	(-0.4, 14.4)	(0.5, 1.3)	(1.0, 12.3)	(1.8, 5.8)
1.36	(-0.7, 9.8)	(0.3, -0.8)	(0.7, 17.8)	(1.4, 5.6)
1.34	(-1.1, 8.9)	(-0.0, -5.1)	(0.85, 117.3)	(1.1, 5.3)
1.32	(-1.3, 7.7)	(**, **)	(-0.1, -1.8)	(0.8, 4.6)
1.30	(-1.7, 8.0)	(-1.6, 57.5)	(-0.4, 2.9)	(0.5, 2.9)
1.28	(-1.9, 7.2)	(-1.3, 21.0)	(-0.7, 3.8)	(0.2, 0.8)
1.26	(-2.2, 7.1)	(-1.5, 14.7)	(-0.9, 4.0)	(-0.0, -8.3)

However this error does not take full account of the effects of the discrepancy between the inclusive measurement of $R(s')$ and the sum of the exclusive channels in the energy region $\sqrt{s'} \lesssim 2\text{ GeV}$. This discrepancy in $R(s')$ leads, on its own, to a difference of 2.3×10^{-4} in the value of $\Delta\alpha_{\text{had}}(M_Z^2)$; see Table 2. Clearly it is important to resolve the dilemma.

We confirm the general conclusion of Jegerlehner [5] that analytic continuation does not improve the accuracy of the determination $\Delta\alpha_{\text{had}}(M_Z^2)$. We find the evaluation of $\Delta\alpha_{\text{had}}(s)$ at the space-like value $s = -s_0 = -6\text{ GeV}^2$ has a reduced error of $\pm 1.4 \times 10^{-4}$. However the reduction in the error is more than offset by the uncertainty in the perturbative QCD analytic continuation from $s = -s_0$ to $s = -M_Z^2$, which arises from its dependence on the choice of charm mass and of the QCD scale.

On the other hand the evaluation of (4) at different space-like values $s = -s_0$ proves to be very informative.

Table 9. The pole mass of the charm quark determined from demanding self-consistency of the space-like evaluation of $\Delta\alpha_{\text{had}}$ (that is requiring the discrepancy $d(s_0) \simeq 0$ for all s_0), compared to the PDG value [12]. Inclusive (exclusive) mean that $R(s')$ is determined from inclusive data (sum of the exclusive channels) in the region $\sqrt{s'} \lesssim 2\text{ GeV}$. In both cases the lower and upper values quoted for m_c correspond to scale choices $\mu = 50$ and 20 GeV respectively

Source	$m_c(\text{GeV})$
inclusive	1.33–1.40
exclusive	1.26–1.34
PDG	1.46 ± 0.11

For each evaluation $\Delta\alpha_{\text{had}}(s_0)$ at a different, but sufficiently large, s_0 , we can analytically continue to $s = -M_Z^2$, and then around the semicircle in the complex plane to $s = M_Z^2$, using perturbative QCD. We can compare these determinations of $\Delta\alpha_{\text{had}}(M_Z^2)$ with the traditional method of directly evaluating (4) at $s = M_Z^2$. In fact we found it convenient to study the difference

$$d(s_0) \equiv \Delta\alpha_{\text{had}}(M_Z^2)|_{\text{direct}} - \Delta\alpha_{\text{had}}(M_Z^2)|_{\text{anal. cont. from } s_0} \quad (15)$$

as a function of s_0 . A self-consistent analysis requires $d(s_0) \simeq 0$ for all choices of s_0 .

Indeed we found that the study of $d(s_0)$ sheds light on the ‘inclusive’ versus ‘exclusive’ data dilemma, and provides evidence in favour of the former. But first we noted that the perturbative QCD analytic continuation was sensitive to the pole mass m_c of the charm quark, as well as to the QCD scale μ . To eliminate the dependence on m_c (and m_b) we evaluated $d(s_0)$ using the data for $R(s')$ in the region $2m_\pi < \sqrt{s'} < 2.8\text{ GeV}$ and three-flavour perturbative QCD elsewhere. We performed the analysis using first the inclusive, and then the exclusive, data for $\sqrt{s'} \lesssim 2\text{ GeV}$; in each case for three choices of the QCD scale. We found

the ‘inclusive’ $d(s_0)$ values were more self-consistent than the ‘exclusive’ behaviour of $d(s_0)$.

We exploited the sensitivity of the $d(s_0)$ analysis to the pole mass of the charm quark in order to determine the value of m_c . To do this we repeated the above procedure with the charm data reinstated and used four-flavour QCD. If the ‘inclusive’ data are used, we found that indeed there is a unique value of m_c for which we obtain the same $\Delta\alpha_{\text{had}}(M_Z^2)$ for the different space-like $s = -s_0$ values and for the direct evaluation at $s = M_Z^2$. In this way, we determine the pole mass to be

$$m_c = 1.33 - 1.40 \text{ GeV}, \quad (16)$$

if the QCD scale is $\mu = 50$ or 20 GeV respectively. Just as in the three-flavour study, we found that the four-flavour analysis is less consistent if the ‘exclusive’ data choice is employed.

In summary, we have presented quite a body of evidence to show that self-consistency of the results for the space-like and time-like evaluation of dispersion relation (4) selects the inclusive measurements of $R(s')$ in the region $1.46 < \sqrt{s'} < 1.9$ GeV, as compared to the values of $R(s')$ deduced from the sum of the exclusive channels. Thus we conclude that

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (274.26 \pm 1.90) \times 10^{-4}, \quad (17)$$

and hence that

$$\alpha^{-1}(M_Z^2) = 128.972 \pm 0.026. \quad (18)$$

The corresponding results using the exclusive data, which are not favoured, are $(276.49 \pm 2.14) \times 10^{-4}$ and 128.941 ± 0.029 . Precise measurements of $R(s')$ in the energy region $\sqrt{s'} \lesssim 2$ GeV are necessary to confirm our conclusion and, more important, to improve the precision in the determination of the QED coupling on the Z pole.

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Note added in Proof: The final BES measurements have just become available, see [25]. The measurements of R at $\sqrt{s} = 2, 2.2, 2.4$ and 2.5 GeV are slightly higher than the preliminary measurements [2]. In fact the latter three points now lie on our input curve for R that is shown in Fig. 1. The point at $\sqrt{s} = 2$ GeV has increased by about 5% to $R = 2.18 \pm 0.07 \pm 0.18$ [25]. These small changes do not affect the results presented in this paper.